

## A SIMPLE, FIRST STEP TO THE OPTIMIZATION OF REGENERATOR GEOMETRY\*

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This paper presents a simplified set of equations for calculating and optimizing regenerator geometries. A number of competing parameters must be accounted for in the design of a regenerator. To obtain high values of effectiveness, there is need for a large surface area for heat transfer and high heat capacity. The void volume should be small to maintain the pressure ratio in the entire system. Moreover, the pressure drop should be small compared to the absolute pressure. With the assumptions made here, the calculations can be done with a hand calculator.

The equations show that the optimum regenerator length, hydraulic radius, and porosity are independent of mass flow rate and the gas cross-sectional area is proportional to the mass flow rate. It is shown that using gas gaps between parallel plates produces a significantly better regenerator than is possible with packed spheres or screens, particularly at temperatures below approximately 50 K. The improvement is due to enhanced heat transfer and the flexibility in using a lower porosity in the gap configuration.

Key Words: Cryocoolers, cryogenics, gaps, geometry, heat exchangers, optimization, packed spheres, regenerators.

## 1. Introduction

The regenerator is often one of the major loss sources in regenerative-cycle refrigerators. It contributes loss terms due to its limited heat transfer units, limited matrix specific heat, pressure drop, dead volume, and axial thermal conduction. An optimum design of these regenerators would significantly improve the performance of the overall refrigerator.

The calculation of regenerator performance with various assumptions has been discussed extensively in the literature. First order calculations assume the void volume in the regenerator is zero. Such calculations were first done by Lambertson [1] and Hausen [2] with graphs and tables presented in the books by Kays and London [3] and Schmidt and Willmott [4]. Second order calculations take into account the regenerator void volume but neglect any pressure oscillation in the void volume gas. These calculations have been done by Heggs and Carpenter [5], by Daney and Radebaugh [6], and by others cited in these two works. A third order calculation considers void volume in the regenerator and a pressure oscillation as seen by regenerators in regenerative-cycle refrigerators. Gary, et al. [7] discuss such a third order calculation where very few assumptions are made.

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All of the cited calculations require the regenerator geometry as the input parameter and compute the regenerator effectiveness. The problem facing the designer is how to determine the optimum geometry without resorting to trial and error through such performance calculations. Such an approach can be very costly and time consuming, especially for third order calculations where considerable computer time is required for each case. In this paper we discuss a simple technique for determining the optimum geometry associated with a desired value of regenerator effectiveness. The technique reverses the "normal" first order calculations and computes the various geometrical parameters that give the minimum regenerator void volume. If the void volume is not negligible a second iteration should be done with reversed second or third order calculations.

## 2. Basis for optimization

In any optimization scheme it is important to determine the proper parameter to be optimized. Most often the input power is minimized for a refrigerator with a specified net refrigeration power. (In some applications, such as cooling SQUID's, the net refrigeration power is zero and the gross refrigeration is used entirely for loss terms. Nevertheless, the input power is still the quantity to be minimized in SQUID cryocoolers.) It would appear that a general system requirement would preclude a simple optimization of a single component like a regenerator, especially when the regenerator may consist of several stages. Any exact analysis must consider the entire system as a coupled system and various loss terms will necessarily be dependent on each other. However, Smith and coworkers [8-11] have found that a decoupled approach to the loss terms has resulted in relatively good approximations to the overall system performance. In the decoupled approximation the gross refrigeration power,  $\dot{Q}_r$ , is absorbed by several independent terms:

$$\dot{Q}_r = \dot{Q}_{net} + \dot{Q}_{rad} + \dot{Q}_c + \dot{Q}_{reg} + \dot{Q}_s + \dot{Q}_h + \dots, \quad (1)$$

where  $\dot{Q}_{net}$  is the net refrigeration power,  $\dot{Q}_{rad}$  is the radiation loss,  $\dot{Q}_c$  is the conduction loss,  $\dot{Q}_{reg}$  is the loss due to regenerator ineffectiveness,  $\dot{Q}_s$  is the shuttle heat loss due to the oscillating displacer, and  $\dot{Q}_h$  is the loss caused by an excess enthalpy flow through the regenerator during the hot blow for certain conditions with a non-ideal gas. Another possible term would be one due to the pressure drop,  $\Delta P$ , in the system. That term would be used whenever  $\dot{Q}_r$  is calculated from the pressure amplitude at the compressor. If  $\dot{Q}_r$  is calculated from the pressure amplitude at the expansion space, then no such term would be used, but the pressure drop would indicate the required pressure amplitude at the compressor.

In the decoupled approximation of eq.(1) the entire system is optimized by minimizing  $\dot{Q}_r$  for a fixed value of  $\dot{Q}_{net}$ . For an ideal gas  $\dot{Q}_h = 0$ . For the second and third stages  $\dot{Q}_{rad}$  is often negligible. The other terms,  $\dot{Q}_c$ ,  $\dot{Q}_{reg}$ , and  $\dot{Q}_s$ , are then usually of comparable values when a minimum  $\dot{Q}_r$  is found. Likewise, the percentage loss due to a pressure drop,  $\Delta P/P$ , will be comparable to the percentage loss of these other three terms. The simplest approximation for  $\dot{Q}_r$  assumes an isothermal expansion. If an adiabatic expansion is considered,  $\dot{Q}_r$  is reduced by approximately a factor of two [12].

In dimensionless form eq.(1) is given as

$$\dot{Q}_{\text{net}}/\dot{Q}_r + \dot{Q}_{\text{rad}}/\dot{Q}_r + \dot{Q}_c/\dot{Q}_r + \dot{Q}_{\text{reg}}/\dot{Q}_r + \dot{Q}_s/\dot{Q}_r + \dot{Q}_h/\dot{Q}_r + \dots = 1 \quad (2)$$

For any particular refrigerator an experienced designer can then make reasonable estimates of such terms as  $\dot{Q}_c/\dot{Q}_r$  and  $\dot{Q}_{\text{reg}}/\dot{Q}_r$  and  $\Delta P/P$ . For example, with an ideal gas in a cryocooler where  $\dot{Q}_{\text{net}}/\dot{Q}_r$  is to be small, the aforementioned terms may be approximately 0.1-0.2. Good estimates of these terms are the first and most important step in the optimization of a regenerator. The performance of the overall system is also affected by the void volume of the regenerator,  $V_{\text{rg}}$ . A normalized void volume would be  $V_{\text{rg}}/V_e$ , where  $V_e$  is the expansion space volume. The four terms  $\dot{Q}_{\text{reg}}/\dot{Q}_r$ ,  $\dot{Q}_c/\dot{Q}_r$ ,  $\Delta P/P$ , and  $V_{\text{rg}}/V_e$  are interrelated; a decrease of one must cause an increase of another.

The basis for the optimization procedure described herein is to use  $\dot{Q}_{\text{reg}}/\dot{Q}_r$ ,  $\dot{Q}_c/\dot{Q}_r$ , and  $\Delta P/P$  as known input parameters and then find the geometry which gives a minimum  $V_{\text{rg}}/V_e$ . At an early, intermediate point, the regenerator ineffectiveness associated with  $\dot{Q}_{\text{reg}}/\dot{Q}_r$  is calculated. If  $V_{\text{rg}}/V_e \ll 1$ , then the input parameters could be decreased to the point where  $V_{\text{rg}}/V_e$  becomes significant. If  $V_{\text{rg}}/V_e \gg 1$ , then the three input parameters must be increased. If the values are as large as they can be, the process probably is not feasible with the conditions chosen, unless an abnormally large compressor is used. The optimization done here is not fully rigorous since estimates are used for the input parameters and the effects of  $\Delta P/P$  and  $V_{\text{rg}}/V_e$  on the system are not fully quantified.

The geometry of a regenerator can be divided into two components: (a) the configuration of the packing, such as packed spheres, plates, tubes, etc. and (b) the macroscopic geometry, such as cross-sectional area, porosity, length, and hydraulic radius. The optimization procedure discussed here treats both components. The best configuration is the one which has the highest heat transfer rate for a fixed pressure drop. However, any configuration may be chosen and the optimization procedure then gives the best geometrical parameters. Pressure drop and heat transfer correlations often incorporate the friction factor,  $f$ , and the Stanton number

$$N_{\text{st}} = h/(\dot{m}/A_g)c_p \quad (3)$$

where  $h$  is the heat transfer coefficient,  $\dot{m}$  is the mass flow rate,  $A_g$  is the gas cross-sectional area, and  $c_p$  is the specific heat of the gas at constant pressure.

Figure 1 shows a plot of  $N_{\text{st}} N_{\text{pr}}^{2/3}$  and  $f$  as a function of the Reynolds number,  $N_r$ , for one particular configuration, an infinitely long and wide gap. The two curves of  $N_{\text{st}} N_{\text{pr}}^{2/3}$  are for the case of constant heat flow and for constant temperature. Kays and London [3] provide the correlations for gaps and several other configurations. The Prandtl number  $N_{\text{pr}}$  is included to make the correlations valid for a wide range of fluids. For helium gas  $N_{\text{pr}} = 0.666$  under ideal conditions. Note that both  $N_{\text{st}} N_{\text{pr}}^{2/3}$  and  $f$  have nearly the same dependence on  $N_r$ . For regenerators the curve of interest is that of constant heat flow. The practical difficulty in

achieving a uniform hydraulic radius, however, suggests the lower, constant temperature curve may be more realistic.

We define the ratio:

$$\alpha = N_{st} N_{pr}^{2/3} / f \quad (4)$$

A high value of  $\alpha$  is desired since it gives a high heat transfer for a given pressure drop. (A more quantitative argument is given later.) Figure 2 shows  $\alpha$  as a function of the Reynolds number for several packing configurations and for a constant temperature heat transfer. The data are from Kays and London [3]. The two important points to note from these curves are (a) they are nearly independent of Reynolds number, and (b) a gap configuration has the highest  $\alpha$  - significantly higher than that of a packed bed of spheres. Because  $\alpha$  is nearly independent of Reynolds number, it becomes a useful parameter to use in calculations since there is no need to specify the flow rate.

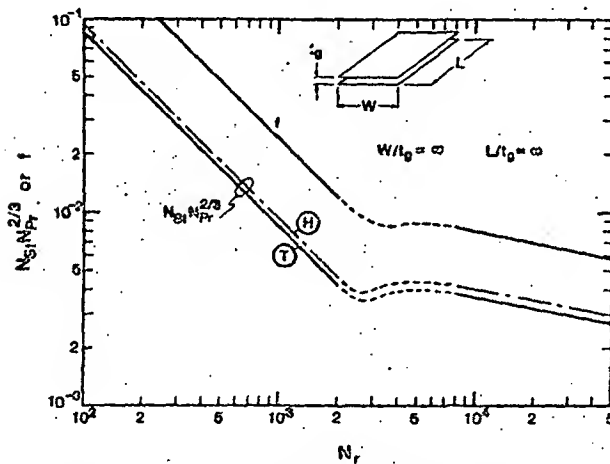


Figure 1. Heat transfer and friction factor curves as a function of Reynolds number for gas flow in a gap.

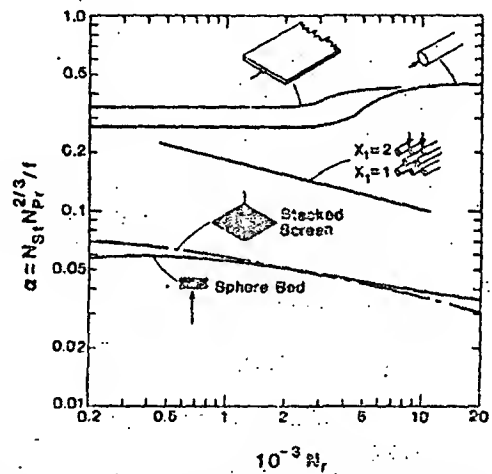


Figure 2. Ratio of Heat transfer and friction factor curves as a function of Reynolds number for several configurations.

### 3. Optimization procedure

The equations which are developed in the following sections describe the procedure to calculate the actual dimensions of an optimum regenerator geometry. The development begins by relating  $Q_c/Q_r$  to a regenerator ineffectiveness. By using published performance curves for regenerators, a set of values for the number of heat transfer units per half cycle,  $N_{tu}$ , and the ratio of matrix heat capacity to the heat capacity of the fluid that passes through the regenerator,  $C_r/C_f$ , can be found that yields the correct ineffectiveness. Because an infinite set of values will yield the same ineffectiveness, the goal is to find the set of  $N_{tu}$  and  $C_r/C_f$  values that gives the minimum  $V_{rg}/V_e$ . Expressions for the geometrical parameters of cross-sectional area, length, porosity, and hydraulic radius are developed in terms of the optimized  $N_{tu}$  and  $C_r/C_f$  and the input parameters  $Q_c/Q_r$  and  $\Delta P/P$ . The configurational parameter  $\alpha$  is used in the equations to make them valid for all  $N_r$  and to show how a maximum  $\alpha$  produces a minimum  $V_{rg}/V_e$ .

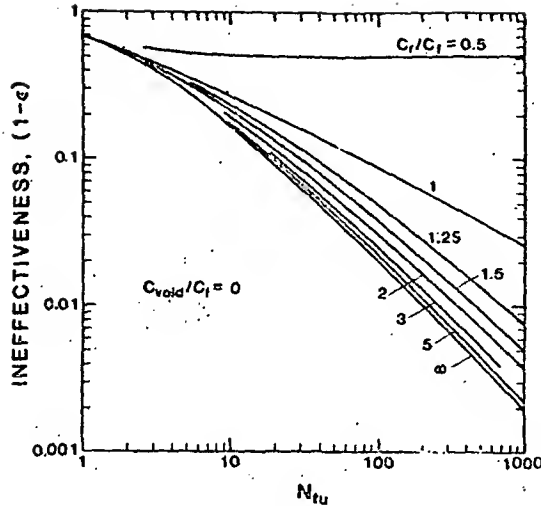


Figure 3. Calculated regenerator ineffectiveness for zero void volume.

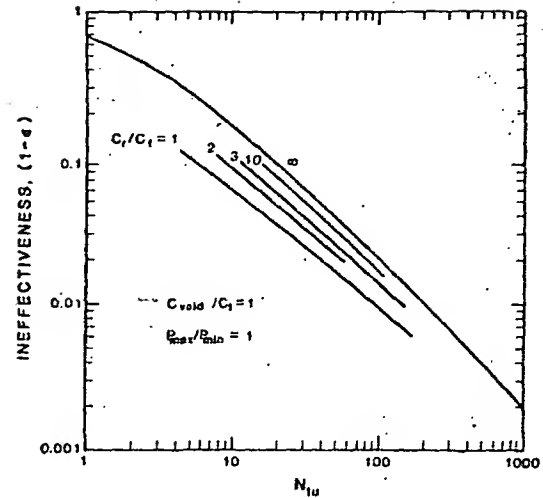


Figure 4. Calculated regenerator ineffectiveness for a finite void volume but with no pressure wave.

### 3.1 Performance curves

The effectiveness,  $\epsilon$ , of a regenerator is defined as the ratio of heat actually transferred to the maximum possible heat transfer. The first optimization step utilizes the performance curves as shown in figures 3 and 4. Figure 3 shows the ineffectiveness,  $1 - \epsilon$ , as a function of  $N_{tu}$  and  $C_r/C_f$ . The plot assumes zero void volume, zero thermal conductivity in the axial direction, infinite thermal conductivity in the radial direction, gas and matrix thermal properties independent of temperature, constant inlet temperatures, constant flow rate, and constant heat transfer coefficient. The curves in figure 3 are plotted from data in Kays and London [3] modified by the appropriate change for our definition of  $N_{tu}$ . Figure 4 is for the case of a finite void volume without a pressure wave [5,6]. All other assumptions are the same as for figure 3. Other curves could be generated using fewer assumptions. Since the thermal conductivity in the axial direction is taken into account separately via a decoupled loss term, it should always be neglected in the performance curves.

For a predetermined value of  $\dot{Q}_{reg}/\dot{Q}_r$  a value of  $1 - \epsilon$  can be calculated for a regenerator operating between some upper temperature  $T_U$  and a lower temperature  $T_L$ . By definition

$$1 - \epsilon = \dot{Q}_{reg}/\dot{m}(H_U - H_L), \quad (5)$$

where  $H$  is the enthalpy per unit mass. Since refrigeration in a regenerative-cycle cryocooler occurs only during one half-cycle, the refrigeration rate is given as

$$\dot{Q}_r = \dot{m}q_r/2, \quad (6)$$

where  $q_r$  is the heat absorbed per unit mass during the expansion process. Combining eqs. (5) and (6) gives

$$1 - \varepsilon = (\dot{Q}_{reg}/\dot{Q}_r)q_r/2(H_U - H_L). \quad (7)$$

As shown in figure 3, a given ineffectiveness is satisfied by a wide range of  $N_{tu}$  and  $C_r/C_f$  values. The optimum combination will become evident later, but we proceed with the optimization procedure assuming that the best  $N_{tu}$  and  $C_r/C_f$  have already been determined.

### 3.2 Gas cross-sectional area, $A_g$

A high  $N_{tu}$  in a heat exchanger is achieved by using a high surface area, but the high surface area will also lead to a large pressure drop unless the cross-sectional area is sufficiently large. The pressure drop in the core of the regenerator is given by

$$\Delta P = f(\dot{m}/A_g)^2 L/2\rho r_h, \quad (8)$$

where  $f$  is the friction factor,  $L$  is the regenerator length,  $\rho$  is the gas density, and  $r_h$  is the hydraulic radius. Equation (8) does not consider the entrance and exit pressure changes associated with acceleration and deceleration, but these terms are generally negligible in a well designed regenerator. By using the term  $\alpha$  from eq. (4), the pressure drop in eq. (8) can be related to the heat transfer by

$$\Delta P = N_{st,pr}^{2/3} (\dot{m}/A_g)^2 L/2\alpha\rho r_h. \quad (9)$$

By using the definition for  $N_{st}$  in eq. (3) and the following definitions

$$N_{tu} = hA/\dot{m}c_p \quad (10)$$

$$r_h = LA_g/A, \quad (11)$$

where  $A$  is the total heat transfer area, we can express  $N_{tu}$  in terms of  $N_{st}$  as

$$N_{tu} = N_{st}L/r_h. \quad (12)$$

Rearranging eq. (12) and substituting into eq. (9) for  $N_{st}$  allows us to write for the gas cross-sectional area

$$A_g/\dot{m} = [N_{tu} N_{pr}^{2/3} / 2\alpha\rho\Delta P]^{1/2}. \quad (13)$$

Equation (13) is one of the most important equations in the design of a heat exchanger, whether it is regenerative or recuperative. It gives the gas cross-sectional area needed to achieve a given  $N_{tu}$  for a fixed value of  $\Delta P$  across the exchanger. Equation (13) also shows that  $A_g$  should be linearly proportional to  $\dot{m}$  and that a large  $\alpha$  means a small  $A_g$ . The term  $\alpha$  is nearly constant for a given heat exchanger packing and is taken from figure 2. An average value of  $\rho$  must be used when large temperature differences occur from one end to the other. In dealing with a mass flow rate that is not constant, e.g. sinusoidal flow, the average  $\dot{m}$  should be used to determine  $A_g$ . Likewise the  $\Delta P$  then corresponds to the average  $\dot{m}$  and not the peak.

### 3.3 Regenerator length, L

Equation (13) shows that  $V_{rg}$  could be made as small as desired by using a short regenerator. However conduction in the regenerator would then become large. We now calculate the length associated with the given value of  $\dot{Q}_c/\dot{Q}_r$ . By definition

$$\dot{Q}_c = (A_m/L) \int_{T_L}^{T_U} k dT, \quad (14)$$

where  $A_m$  is the regenerator solid or matrix cross-sectional area and  $k$  is its thermal conductivity. The term  $A_m$  is related to  $A_g$  through the porosity  $n_g$  by

$$n_g = A_g / (A_m + A_g). \quad (15)$$

Equations (15) and (6) are substituted into eq. (14) to yield

$$L = \frac{2(A_g/\dot{m})(1 - n_g) \int k dT}{n_g q_r (\dot{Q}_c/\dot{Q}_r)}. \quad (16)$$

For some packing configurations  $n_g$  is fixed, such as packed spheres or stacked screens. In those cases,  $L$  can then be calculated from the known input parameter  $\dot{Q}_c/\dot{Q}_r$ . For other configurations, such as gaps, the porosity is entirely flexible and  $L$  cannot be calculated until the porosity is first calculated. Note that eq. (16) also applies to either recuperative or regenerative heat exchangers, since the  $C_r/C_f$  term has not been used.

### 3.4 Porosity, $n_g$

In configurations where the porosity is flexible, additional matrix heat capacity can be obtained by adding more matrix material and still keeping  $A_g$  constant. However, at some point conduction effects become important. The regenerator matrix heat capacity is given by

$$C_r = V_m \rho_m c_m \quad (17)$$

where  $\rho_m$  is the matrix density,  $c_m$  is the matrix specific heat, and  $V_m$  is the matrix volume

$$V_m = A_m L \quad (18)$$

Equation (15) is used to express  $V_m$  in terms of  $A_g$  by

$$V_m = A_g L (1 - n_g) / n_g \quad (19)$$

Substitution of eq. (19) into eq. (17) gives

$$C_r = \rho_m c_m A_g L (1 - n_g) / n_g \quad (20)$$

The heat capacity of the fluid that passes through the regenerator is

$$C_f = \dot{m} c_p \tau / 2 = \dot{m} c_p / 2v \quad (21)$$

where  $\tau$  is the period of one cycle and  $v$  is the frequency of operation for the regenerative cryo-cooler. The use of eq. (21) now restricts subsequent results to regenerative heat exchangers. When eq. (20) is divided by eq. (21) and rearranged the result is

$$n_g / (1 - n_g) = 2v(\rho_m c_m)(A_g / \dot{m})L / c_p (C_r / C_f) \quad (22)$$

For simplicity in subsequent calculations we define  $X$  as



$$X = n_g / (1 - n_g) \quad (23)$$

or

$$n_g = X / (1 + X) \quad (24)$$

The length in eq. (16) is substituted into eq. (22) to yield

$$X = 2(A_g / \dot{m}) \left[ \frac{v(\rho_m c_m) \int k dT}{c_p q_r (C_r / C_f) (\dot{Q}_c / \dot{Q}_r)} \right]^{1/2} \quad (25)$$

In calculating  $n_g$  the ratio  $C_r / C_f$  is assumed to be known from the performance curves in figure 3, although any combination of  $H_{tu}$  and  $C_r / C_f$  can be used. The optimum combination is yet to be determined. Equations (24) and (25) show that the porosity is independent of the refrigerator capacity since  $(A_g / \dot{m})$  and the other terms are independent of mass flow rate. Likewise the length in eq. (16) is independent of mass flow rate or refrigerator capacity. Also, the porosity equation applies to the case of variable porosity. The proper technique for dealing with a fixed porosity is discussed in section 5.

### 3.5 Regenerator gas volume, $V_{rg}$

With  $A_g$  and  $L$  already determined,  $V_{rg}$  is easily calculated. However the ratio  $V_{rg} / V_e$  needs to be calculated to determine what effect  $V_{rg}$  will have on the system. The volume of gas in the regenerator void space can be given by

$$V_{rg} = X V_m \quad (26)$$

The matrix volume can be expressed as

$$V_m = C_r / \rho_m c_m \quad (27)$$

which means eq. (26) becomes

$$V_{rg} = X C_r / \rho_m c_m \quad (28)$$

The expansion space volume is given by

$$V_e = C_f / \rho_e c_p, \quad (29)$$

where  $\rho_e$  is the density of gas in the expansion space. The term  $\rho_e$  should be evaluated at the temperature and pressure of the expansion space at the time of flow reversal. For a first approximation the lower temperature  $T_L$  and the average pressure are used. From eqs. (28) and (29) the volume ratio becomes

$$V_{rg}/V_e = X(C_r/C_f) \rho_e c_p / (\rho_m c_m) \quad (30)$$

This simple expression for  $V_{rg}/V_e$  is useful when  $X$  has already been calculated. However, since the optimum  $N_{tu}$  and  $C_r/C_f$  have not been selected, we now develop an expression, for  $V_{rg}/V_e$  in terms of  $N_{tu}$  and  $C_r/C_f$ . We substitute eq. (25) for  $X$  into eq. (30), and at the same time eq. (13) is used for  $A_g/\dot{m}$  in the expression for  $X$ . The resultant volume ratio becomes

$$V_{rg}/V_e = p_e \left[ \frac{2N_{tu}(C_r/C_f)N_{pr}^{2/3} v_c f k d T}{\alpha \rho \Delta P q_r (\dot{Q}_c/\dot{Q}_r) (\rho_m c_m)} \right]^{1/2} \quad (31)$$

It now becomes clear how  $V_{rg}/V_e$  depends on the various input and material parameters and the packing configuration. The volume ratio is minimized when the configurational parameter  $\alpha$  is largest. Figure 2 shows that gaps offer the largest  $\alpha$  for the configurations compared. The authors are unaware of other configurations with a higher  $\alpha$ . Of course, the equations derived here are valid if one chooses to use a configuration with a smaller  $\alpha$ . In regard to matrix material parameters, the ratio  $[f k d T / (\rho_m c_m)]^{1/2}$  should be minimized. The input parameters  $\Delta P/P$  and  $\dot{Q}_c/\dot{Q}_r$  should be made large to reduce  $V_{rg}/V_e$  but they cannot be made larger than about 0.1-0.2.

A minimum value for  $V_{rg}/V_e$  is obtained when the product  $N_{tu}(C_r/C_f)$  is a minimum. Figure 5 shows curves of  $N_{tu}(C_r/C_f)$  vs.  $(C_r/C_f)$  for various values of ineffectiveness. We see from figure 5 that  $C_r/C_f = 1.8$  gives a minimum in  $N_{tu}(C_r/C_f)$  for all values of  $1 - \epsilon$  shown. The curves of figure 5 are subject to some uncertainty since they were derived from the graph in figure 3 rather than using numerical calculations. Fortunately the minimum in  $N_{tu}(C_r/C_f)$  is fairly broad so some uncertainty can be tolerated. The calculation of  $(A_g/\dot{m})$ ,  $L$ ,  $n_g$ , and  $V_{rg}/V_e$  is now done using the optimum  $N_{tu}$  and  $C_r/C_f$  combination from figure 5.

For an ideal gas and isothermal expansion we have

$$p = P/RT \quad (32)$$

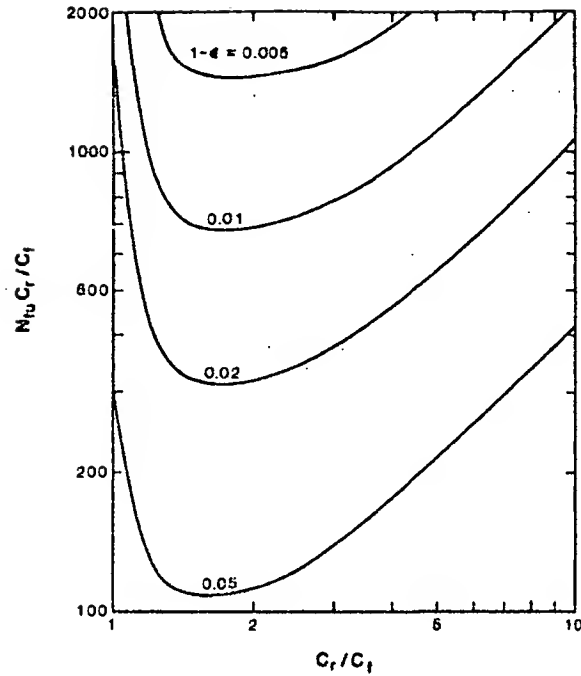


Figure 5. The  $N_{tu}(C_r/C_f)$  product for several values of regenerator ineffectiveness from the case of zero void volume.

$$\rho_e = P/RT_e \quad (33)$$

$$q_r = RT_e \ln(P_h/P_l), \quad (34)$$

where  $R$  is the gas constant,  $P$  is the average pressure,  $P_h$  is the high pressure,  $P_l$  is the low pressure, and  $T_e$  is the expansion space temperature. (Usually  $T_e = T_L$ ). Then eq. (31) becomes

$$V_{rg}/V_e = \frac{1}{RT_e} \left[ \frac{2N_{tu}(C_r/C_f)^{2/3} N_{pr} c_p T_e f k d T}{T_e \alpha (\rho_m c_m) (\dot{Q}_c/\dot{Q}_r) (\Delta P/P) \ln(P_h/P_l)} \right]^{1/2} \quad (35)$$

for an ideal gas and isothermal expansion.

Equation (35) shows that the volume ratio is independent of pressure and will increase as the temperature  $T_e$  is lowered. For the optimum  $(C_r/C_f) = 1.8$ , the ineffectiveness is approximately

$$1 - \epsilon = 2.1 N_{tu}^{-0.9}, \quad (36)$$

for  $N_{tu} > 50$ . The term  $N_{tu}$  is then proportional to  $(1 - \epsilon)^{-1.1}$  and using eq. (7) for  $(1 - \epsilon)$  shows  $N_{tu}$  is proportional to  $(\dot{Q}_{reg}/\dot{Q}_r)^{-1.1}$ . The denominator inside the brackets of eqs. (31) and (35) subsequently contains the three input parameters in the form of  $(\dot{Q}_{reg}/\dot{Q}_r)^{1.1}(\dot{Q}_c/\dot{Q}_r)(\Delta P/P)$ . Such an expression would suggest that  $(\dot{Q}_{reg}/\dot{Q}_r)$  should be larger than the other terms for minimizing  $V_{rg}/V_e$ .

There may be instances where the total regenerator volume is to be minimized. This possibility may occur when there are external packaging constraints or in cases where no pressure wave exists. For instance, regenerators used in magnetic refrigerators operate at constant pressure. Here, the void volume is often of little concern. Of more importance is the total regenerator volume, since it may need to fit inside a magnetic field region. The total regenerator volume,  $V_{rt}$ , can be expressed in reduced units by

$$V_{rt}/V_e = V_{rg}/V_e n_g \quad (37)$$

The use of eqs. (24), (25), and (30) leads to

$$V_{rt}/V_e = \frac{\rho_e (C_r/C_f) c_p}{(\rho_m c_m)} + \rho_e \left[ \frac{2N_{tu} (C_r/C_f) N_{pr}^{2/3} v c_p \int k dT}{\alpha p \Delta P \eta_r (\dot{Q}_c/\dot{Q}_r) (\rho_m c_m)} \right]^{1/2} \quad (38)$$

The first term on the right hand side of eq. (38) is the reduced matrix volume and the second term is  $V_{rg}/V_e$  of eq. (31). It is now apparent that the minimum in  $V_{rt}/V_e$  occurs at a smaller value of  $(C_r/C_f)$  than the 1.8 used for  $V_{rg}/V_e$ , and the  $N_{tu}$  will be higher than for that case.

### 3.6 Hydraulic radius

Here we derive the hydraulic radius and other characteristic dimensions of the regenerator packing that must be used to give the desired  $N_{tu}$  in a regenerator with a length and cross-sectional area as previously calculated. When both sides of eq. (12) are multiplied by  $N_{pr}^{2/3}$  and rearranged, the result is

$$N_{st} N_{pr}^{2/3} = N_{tu} N_{pr}^{2/3} r_h / L \quad (39)$$

The Reynolds number is defined as

$$N_r = 4r_h / \mu (A_g/\dot{m}), \quad (40)$$

where  $\mu$  is the gas viscosity.

For every geometry of interest there is some function  $g$  where

$$N_{st} N_{pr}^{2/3} = g(N_r). \quad (41)$$

Equations (39-41) represent three equations in the three unknowns,  $r_h$ ,  $N_r$ , and  $N_{st} N_{pr}^{2/3}$  that must be solved numerically in the general case. When  $g$  is in graphical form, the solution is easily found by trial and error. For a chosen value of  $r_h$ , eq. (39) is used to find  $N_{st} N_{pr}^{2/3}$ . The same value of  $r_h$  gives  $N_r$  from eq. (40) which in turn gives a value of  $N_{st} N_{pr}^{2/3}$  from the graph of eq. (41). If the two values of  $N_{st} N_{pr}^{2/3}$  do not agree, another value of  $r_h$  is chosen until the two  $N_{st} N_{pr}^{2/3}$  values agree.

For laminar flow, eq. (41) is

$$N_{st} N_{pr}^{2/3} = a/N_r, \quad (42)$$

where  $a$  is a constant depending on geometry. For infinitely wide and long gaps with constant temperature heat transfer,  $a = 8.5$ . For infinitely long tubes,  $a = 4.2$ . Equations (39), (40), and (41) are solved then by

$$r_h = \left[ \frac{a \mu (A_g / \dot{m})}{4 N_{tu} N_{pr}^{2/3}} \right]^{1/2} \quad (43)$$

In the high-performance heat exchangers discussed here, the flow is usually laminar. The value of  $\alpha$  in figure 2 can be refined with  $N_r$  if necessary. Since  $\alpha$  is only a weak function of  $N_r$ , further iterations may not be necessary.

The characteristic dimension of a heat exchanger may be expressed in terms of the hydraulic radius. They are given for various geometries as follows:

$$t_g = 2r_h \quad \text{gap thickness} \quad (44)$$

$$d = 4r_h \quad \text{tube diameter} \quad (45)$$

$$d = 6r_h/X \quad \text{sphere diameter} \quad (46)$$

$$d = 4r_h/X \quad \text{wire diameter.} \quad (47)$$

For a gap configuration the thickness of the plates that comprise the matrix is

$$t_m = t_g/X. \quad (48)$$

For all of the preceding calculations to be valid, the thermal penetration depth  $\lambda$  must be large enough to penetrate throughout the matrix. For a semi-infinite plate the thermal penetration depth for sinusoidal heat flow is

$$\lambda = [k/(\rho_m c_m) \pi \nu]^{1/2}. \quad (49)$$

For a set of stacked plates with gas on both sides we require

$$t_m < 2\lambda. \quad (50)$$

### 3.7 Equation summary

As an aid to computations for regenerators, the significant steps and equations are summarized in the order necessary for a regenerator with flexible porosity.

1. Determine largest input parameters  $\dot{Q}_{reg}/\dot{Q}_r$ ,  $\dot{Q}_c/\dot{Q}_r$ ,  $\Delta P/P$  the overall system can tolerate.
2. Evaluate system and material parameters  $\nu$ ,  $T_U$ ,  $T_L$ ,  $n$ ,  $r$ ,  $q_r$ ,  $\rho$ ,  $\rho_e$ ,  $c_p$ ,  $N_{pr}$ ,  $\mu$ ,  $(\rho_m c_m)$ , and  $\int k dT$ .
3. Ineffectiveness:  $1 - \varepsilon = (\dot{Q}_{reg}/\dot{Q}_r) q_r / 2(H_U - H_L)$ .
4. From figure 3 or 5 determine:  $N_{tu}$  at  $C_r/C_f = 1.8$ .
5. Determine packing parameters  $\alpha$  from figure 2 and laminar coefficient  $a$  in eq. (42).
6. Gas cross-sectional area:  $A_g/\dot{m} = [N_{tu} N_{pr}^{2/3} / 2\alpha \rho \Delta P]^{1/2}$ .
7. Porosity:  $n_g = X/(1 + X)$ .

$$X = 2(A_g/\dot{m}) \left[ \frac{\nu(\rho_m c_m) \int k dT}{c_p q_r (C_r/C_f) (\dot{Q}_c \dot{Q}_r)} \right]^{1/2}.$$

$$\text{(in original parameters)} \quad X = 2 \left[ \frac{N_{tu} N_{pr}^{2/3} v(\rho_m c_m) f k d T}{2 \alpha \rho \Delta P c_p q_r (C_r/C_f) (\dot{Q}_c/\dot{Q}_r)} \right]^{1/2} \quad (51)$$

8. Regenerator gas volume:  $V_{rg}/V_e = X(C_r/C_f) c_p \rho_e / (\rho_m c_m)$ .

$$\text{(in original parameters)} \quad V_{rg}/V_e = \rho_e \left[ \frac{2 N_{tu} (C_r/C_f) N_{pr}^{2/3} v c_p f k d T}{\alpha \rho \Delta P q_r (\dot{Q}_c/\dot{Q}_r) (\rho_m/c_m)} \right]^{1/2}$$

9. Regenerator length:  $L = 2(A_g/\dot{m}) f k d T / X q_r (\dot{Q}_c/\dot{Q}_r)$ .

$$\text{(in original parameters)} \quad L = \left[ \frac{c_p (C_r/C_f) f k d T}{v(\rho_m c_m) q_r (\dot{Q}_c/\dot{Q}_r)} \right]^{1/2} \quad (52)$$

10. Hydraulic radius:  $r_h = \left[ \frac{a L \mu (A_g/\dot{m})}{4 N_{tu} N_{pr}^{2/3}} \right]^{1/2}$ , laminar flow.

$$\text{(in original parameters)} \quad r_h = (a \mu)^{1/2} \left[ \frac{c_p (C_r/C_f) f k d T}{2 \alpha \rho \Delta P N_{tu} N_{pr}^{2/3} v(\rho_m c_m) q_r (\dot{Q}_c/\dot{Q}_r)} \right]^{1/2} \quad (53)$$

11. Reynolds number:  $N_r = 4 r_h / \mu (A_g/\dot{m})$   
Recalculate  $r_h$  if flow not laminar.

12. Reevaluate  $\alpha$ . If necessary, repeat from step 6.

13. For gaps:  $t_g = 2 r_h$ ,  $t_m = t_g / X$

14. Thermal penetration depth in plates:  $\lambda = [k / (\rho_m c_m) \pi v]^{1/2}$ ,  
sinusoidal flow.

15. Check to see if:  $t_m < 2\lambda$ .

If not, increase  $v$  to decrease  $t_m$ .

16. Error check: (a)  $\Delta P = f L / 2 r_h (A_g/\dot{m})^2$ ,  
with  $f$  from figure 1 or similar curve.  
Calculated value should equal input value.

(b)  $V_{rg}/V_e = 2(A_g/\dot{m}) L v \rho_e$ ,  
should be same as from step 8.

#### 4. Example solutions

The solution to the design equations can best be illustrated through some examples. In the design of a cryocooler for a SQUID, nonmagnetic and nonmetallic parts need to be used in the cold parts to prevent magnetic noise. The regenerator material chosen for this illustration is G-10 fiberglass epoxy. Even though its volumetric heat capacity is small compared with lead and other regenerator materials, its very low thermal conductivity gives it a desirably small value of  $k/\rho_m c_m$ , the important material parameter which appears in eq. (31). In fact G-10 may have a lower  $k/\rho_m c_m$  than Pb-5% Sb although  $k$  for the latter has not been measured.

The parameters chosen for the example solution are given in table 1.

Table 1. Input parameters for example solutions

$1 - \epsilon = 0.01$ ( $\dot{Q}_{reg}/\dot{Q}_r = 0.22$ )	$T_U = 4T_L$
$\dot{Q}_c/\dot{Q}_r = 0.1$	$\alpha = 0.35$ (gap)
$\Delta P/P = 0.05$	regenerator material: G-10
$P = 0.5$ MPa	fluid: ideal helium gas
$P_h/P_L = 2.0$	isothermal expansion

The temperature  $T_L$  is allowed to vary from 4 K to 100 K to evaluate the temperature dependence of the optimum geometry. The upper temperature  $T_U$  of the regenerator is assumed to be  $4T_L$ , which is typical for cryocoolers. Such a dependence of  $T_U$  on  $T_L$  means that eq. (7) may be written as

$$1 - \epsilon = (\dot{Q}_{reg}/\dot{Q}_r) q_r / 6c_p T_L \quad (54)$$

for the case of an ideal gas. Also, for an ideal gas  $q_r$  from eq. (34) can be substituted into eq. (54), and for the normal case of  $T_e = T_L$ , we obtain

$$1 - \epsilon = (\dot{Q}_{reg}/\dot{Q}_r) R \ln(P_h/P_L) / 6c_p \quad (55)$$

For the input parameters in table 1,  $(\dot{Q}_{reg}/\dot{Q}_r) = 0.22$  from eq. (55).

The material parameters needed in the calculations are  $(\rho_m c_m)$  and  $fkdT$ . Figure 6 shows  $(\rho_m c_m)$  for G-10 fiberglass-epoxy, Pb [13], and GdRh [14]. The alloy Pb + 5% Sb is commonly used as a regenerator material and GdRh is a potential regenerator material because of its high specific heat  $(\rho_m c_m)$  at low temperatures. The thermal conductivity of G-10 is the average of the parallel and perpendicular heat flow directions from the work of Kasen, et al. [15]. Table 2 gives representative values of  $(\rho_m c_m)$ ,  $k$ , and  $fkdT$  used in our calculations.

For the ideal helium gas we use  $c_p = 5.19$  J/g·K,  $R = 2.077$  J/g·K,  $N_{pr} = 0.666$ . The viscosity values from McCarty [16] are nearly independent of pressure over the range we have considered and are approximated by  $\mu = 5.0 \times 10^{-6} + 4.6 \times 10^{-6} T^{0.65}$  g/cm.s over the range of 10-300 K.

Figure 7 shows  $(A_g/\dot{m})$  and the length of the regenerator as a function of  $T_L$  for three different frequencies. The term  $(A_g/\dot{m})$  is independent of frequency. The porosity,  $n_g$ , and the volume ratio,  $V_{rg}/V_e$ , are shown in figure 8.

Note that the optimum porosity is on the order of 0.01 for  $T_L = 5$  K. Such a low porosity allows a large volume of regenerator material to be used to obtain a reasonably high total heat capacity. The volume ratio  $V_{rg}/V_e$  is insignificant at higher temperatures and remains smaller



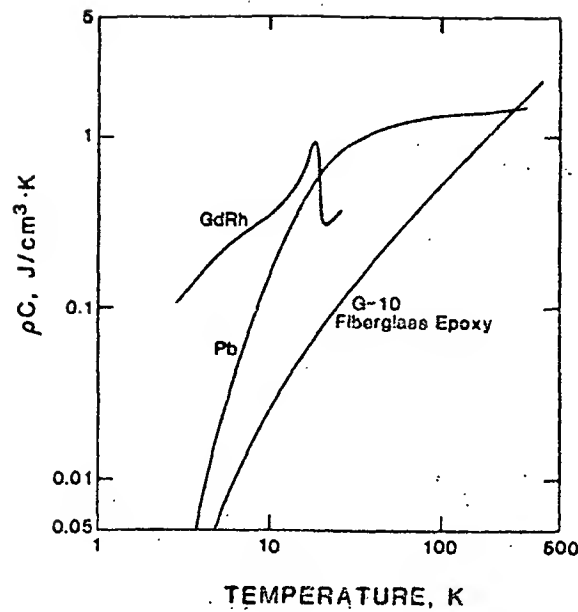


Figure 6. The volumetric heat capacity of candidate regenerator materials.

Table 2. Thermal properties of G-10 fiberglass epoxy

T (K)	$\rho_m c_m$ (J/cm <sup>3</sup> ·K)	k (W/cm·K)	$\int_0^T k dT$ (W/cm)
5	0.0058	$0.815 \times 10^{-3}$	$2.00 \times 10^{-3}$
10	0.0255	$1.24 \times 10^{-3}$	$7.23 \times 10^{-3}$
20	0.078	$1.73 \times 10^{-3}$	$22.3 \times 10^{-3}$
50	0.245	$2.66 \times 10^{-3}$	$89.1 \times 10^{-3}$
100	0.520	$3.76 \times 10^{-3}$	$251. \times 10^{-3}$
200	1.10	$5.53 \times 10^{-3}$	$718. \times 10^{-3}$
300	1.64	$7.17 \times 10^{-3}$	$1355. \times 10^{-3}$

than 1.0 even at  $T_L = 5$  K. For the case of  $\nu = 1$  Hz the volume ratio is small enough that the assumption of zero void volume in the performance curve of figure 3 may hold. For higher frequencies, a more realistic performance curve probably should be used for the lowest temperatures. The small values of  $V_{rg}/V_e$  also imply that the entire process for refrigeration at 4-5 K is feasible with an ideal gas. (Calculations for a real gas are easily done but are not shown here.)

Figure 9 shows the gap and matrix thicknesses as a function of  $T_L$ . Shown for comparison with the matrix thickness,  $t_m$ , is the thermal penetration depth,  $2\lambda$ . For all cases except at 1 Hz and the lowest temperature  $t_m \ll 2\lambda$ . When  $t_m \approx 2\lambda$ , there will be a slight degradation in regenerator performance because the term  $C_r$  will be reduced. The gap thickness has a slight dependence on temperature and is in the range of  $10^{-3}$  cm. An extremely small  $t_g$  could present some practical problems in the construction of the regenerator. The Reynolds number also is very small - in the range of 10-200 - with higher values occurring at lower temperatures and lower frequencies.

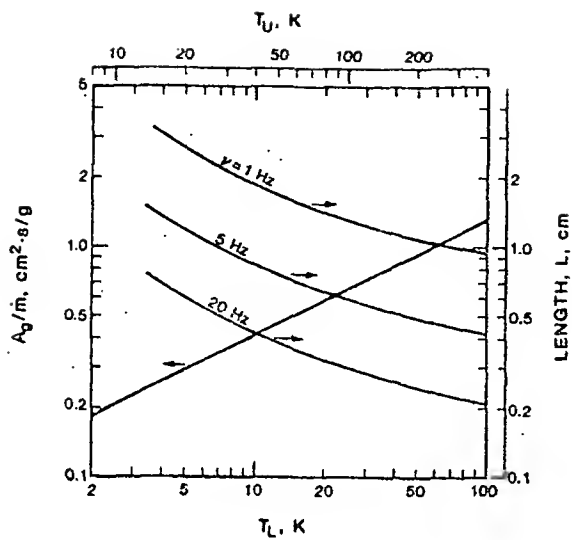


Figure 7. The calculated gas cross-sectional area per unit mass flow, and the regenerator length for the conditions in table 1.

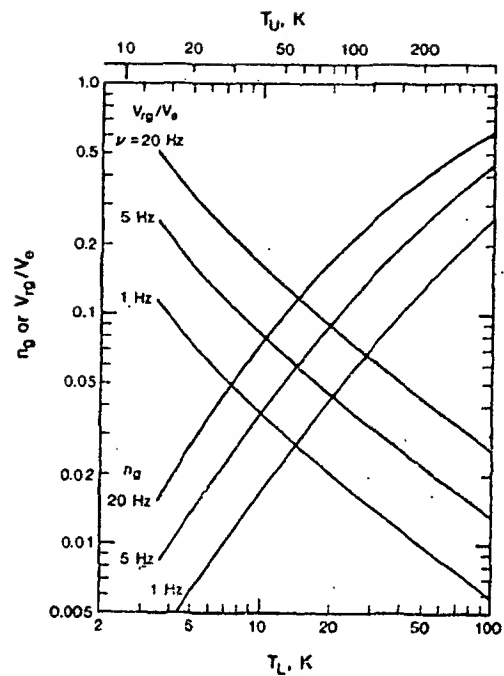


Figure 8. The calculated porosity and the reduced regenerator gas volume for the conditions in table 1.

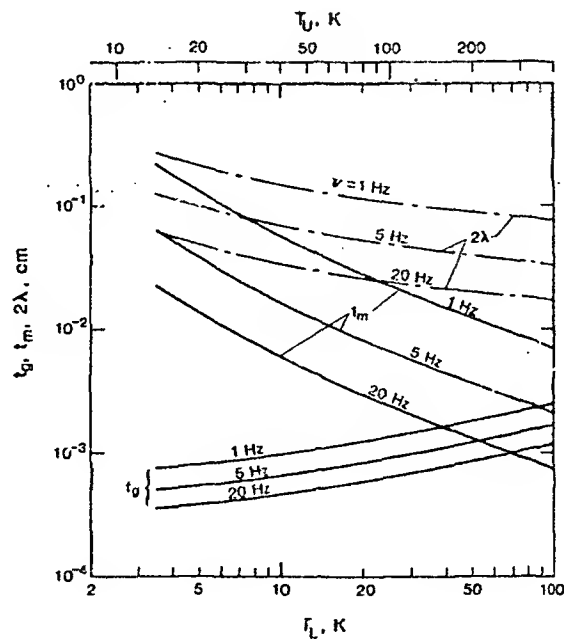


Figure 9. The calculated gap thickness, matrix thickness and thermal penetration depth in the regenerator with conditions in table 1.

## 5. Comparison of gap and packed sphere configurations

Packed spheres are commonly used for regenerator packings since such regenerators are simple to make. However  $\alpha$  is significantly lower for packed spheres than for gaps. Also, the porosity of packed spheres is fixed at about 0.38, which is much higher than optimum for temperatures below about 50 K according to figure 8. In this section we show how much inferior a packed sphere bed is in comparison with the gap configuration. Of course, the packed sphere bed still has the advantage of simplicity. For a fixed  $n_g$ , eq. (51) is solved for the ratio  $N_{tu}/(C_r/C_f)$ . The optimum values for  $N_{tu}$  and  $C_r/C_f$  are found by simultaneous solution with the  $N_{tu}(C_r/C_f)$  curves of figure 5. This approach maintains  $(\dot{Q}_c/\dot{Q}_r)$  at the fixed input of value. Included in the assumptions is that  $fkdT$  for spheres is the same as for the bulk material. After  $N_{tu}$  and  $C_r/C_f$  are determined, the parameters  $A_g/\dot{m}$ ,  $L$ ,  $V_{rg}/V_c$ , and  $r_h$  are calculated from the appropriate equations.

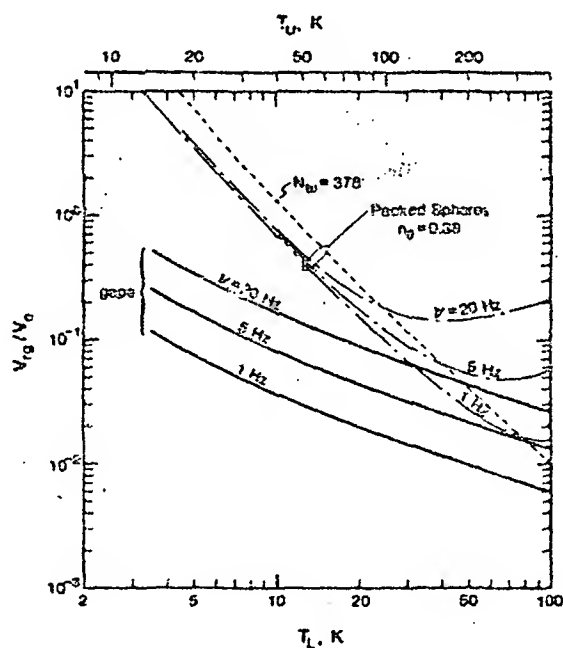


Figure 10. The reduced regenerator gas volume for gaps and for packed spheres. See test for details of packed-sphere curves.

Figure 10 shows  $V_{rg}/V_e$  for both packed spheres of G-10 and gaps between plates of G-10. At higher temperatures the difference between the two sets of curves is mainly due to the higher  $\alpha$  for gaps. At lower temperatures the much smaller  $V_{rg}/V_e$  for gaps is due to both the higher  $\alpha$  and the lower porosity. Also shown in figure 10 for packed spheres is the curve labeled  $N_{tu} = 378$ , which is the value used for the gaps. At higher temperatures this curve gives a lower  $V_{rg}/V_e$  because  $\dot{Q}_c/\dot{Q}_r$  is greater than 0.1. At lower temperatures the reverse is true, and  $\dot{Q}_c/\dot{Q}_r$  is less than 0.1. Wherever the packed-sphere curves are below the  $N_{tu} = 378$  curve, the  $N_{tu}$  is greater than 378. In fact, for temperatures less than 15 K the  $N_{tu}$  is rapidly approaching numbers as high as  $10^5$ , a practical impossibility. Even an  $N_{tu}$  of 378 may be difficult to achieve in practice without extreme care in making all flow channels the same size. If a smaller  $fkdT$  is used in the calculations of the packed powder, an even higher  $N_{tu}$  is needed for the optimum condition. However, the reduction in  $V_{rg}/V_e$  is insignificant at temperatures below about 15 K. The  $N_{tu} = 378$  curve should be considered a lower practical limit for the  $V_{rg}/V_e$  of the packed spheres.

## 6. Conclusions

A simple scheme for optimizing regenerators in cryocoolers has been developed which uses the system loss terms,  $\dot{Q}_{reg}/\dot{Q}_r$ ,  $\dot{Q}_c/\dot{Q}_r$ , and  $\Delta P/P$  as input parameters. Any set of regenerator performance curves can be used, although the examples discussed here use the simplest one of zero void volume. A set of equations were developed to determine the regenerator geometry that yields the minimum void volume. In comparison with other packing configurations it is shown that gaps between parallel plates is the best, producing regenerator dead volumes one to two orders of magnitude smaller for temperatures in the range of 5-10 K. It is shown that G-10 fiberglass-epoxy is a favorable material for a multiple gap regenerator because of its relatively low  $k/\rho_m c_m$  ratio.

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## 8. Nomenclature

English letter symbols		SI units
A	regenerator total heat transfer area	m <sup>2</sup>
A <sub>g</sub>	gas cross-sectional area	m <sup>2</sup>
A <sub>m</sub>	matrix cross-sectional area	m <sup>2</sup>
a	geometrical constant for laminar flow	
C <sub>f</sub>	heat capacity of fluid passed through regenerator	J/K
C <sub>r</sub>	heat capacity of regenerator matrix	J/K
C <sub>void</sub>	heat capacity of gas in regenerator void volume	J/K
c <sub>m</sub>	specific heat of matrix	J/kg·K
c <sub>p</sub>	specific heat at constant pressure of gas	J/kg·K
d	tube, sphere, or wire diameter	m
f	friction factor	
H <sub>U</sub> , H <sub>L</sub>	specific enthalpies at T <sub>U</sub> and T <sub>L</sub>	J/kg
h	heat transfer coefficient	W/m <sup>2</sup> ·K
k	thermal conductivity of matrix	W/m·K
L	length of regenerator	m
ṁ	mass flow rate of gas through regenerator	kg/s

$N_{pr}$	Prandtl number, a fluid properties modulus	
$N_r$	Reynolds number, a flow modulus	
$N_{st}$	Stanton number, a heat transfer modulus	
$N_{tu}$	Number of heat transfer units per half cycle	
$n_g$	porosity of regenerator = $A_g/(A_g + A_m)$	
$P$	average pressure	Pa
$P_h, P_L$	high and low pressures	Pa
$\Delta P$	pressure drop in regenerator	Pa
$\dot{Q}_c$	conduction heat loss	W
$\dot{Q}_h$	enthalpy deficit loss	W
$\dot{Q}_{net}$	net refrigeration power	W
$\dot{Q}_r$	gross refrigeration power	W
$\dot{Q}_{rad}$	radiation heat loss	W
$\dot{Q}_{reg}$	regenerator loss	W
$\dot{Q}_s$	shuttle heat loss	W
$q_r$	heat absorbed per unit mass during expansion	J/kg
$R$	gas constant	J/kg·K
$r_h$	hydraulic radius ( $LA_g/A$ )	m
$T_e$	expansion space temperature	K
$T_U, T_L$	upper and lower temperatures of regenerator	K
$t_g$	gap thickness	m
$t_m$	thickness of plates in gap regenerator	m
$V_e$	maximum expansion space volume	m <sup>3</sup>
$V_m$	matrix volume	m <sup>3</sup>
$V_{rg}$	regenerator void volume	m <sup>3</sup>
$V_{rt}$	regenerator total volume	m <sup>3</sup>
$K$	porosity parameter [ $n_g/(1-n_g)$ ]	

#### Greek letter symbols

$\alpha$	ratio of heat transfer and friction terms ( $N_{st} N_{pr}^{2/3}/f$ )	
$\epsilon$	regenerator effectiveness	
$1-\epsilon$	regenerator ineffectiveness	
$\lambda$	thermal penetration depth in matrix	m
$\mu$	gas viscosity	kg/m·s
$\nu$	cycle frequency	Hz
$\rho$	gas density in regenerator	kg/m <sup>3</sup>
$\rho_e$	gas density in expansion space	kg/m <sup>3</sup>
$\rho_m$	matrix density	kg/m <sup>3</sup>
$\tau$	period of one cycle	s